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Geometric Phase in a Time-Dependent k-Photon Λ-Type Jaynes–Cummings Model

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Abstract By using the Lewis–Riesenfeld invariant theory, we have studied the dynamical and the geometric phases in a generalized time-dependent k-photon Λ-type Jaynes–Cummings model. It is found that, different from the dynamical phases, the geometric phases in a cycle case are independent of the photon numbers, the frequency of the photon field, the coupling coefficient between photons and atoms, and the atom transition frequency.

Keywords Geometric phase · Generalized Jaynes–Cummings model

1 Introduction

It is known that the concept of geometric phase was first introduced by Pancharatnam [1] in studying the interference of classical light in distinct states of polarization. Berry [2] found the quantal counterpart of Pancharatnam's phase in the case of cyclic adiabatic evolution. The extension to non-adiabatic cyclic evolution was developed by Aharonov and Anandan [3]. Samuel and Bhandari [4] generalized the pure state geometric phase further by extending it to non-cyclic evolution and sequential projection measurements. The geometric phase is a consequence of quantum kinematics and is thus independent of the detailed nature of the dynamical origin of the path in state space. This led Mukunda and Simon [5] to put forward a kinematic approach by taking the path traversed in state space as the primary concept for the geometric phase. Further generalizations and refinements, by relaxing the

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conditions of adiabaticity, unitarity, and cyclicity of the evolution, have since been carried out [6]. Recently, the geometric phase of the mixed states has also been studied [7–9].

The quantum invariant theory proposed by Lewis and Riesenfeld [10] is a powerful tool for treating systems with time-dependent Hamiltonians. It was generalized by introducing the concept of basic invariants and used to study the geometric phases [11–14] in connection with the exact solutions of the corresponding time-dependent Schrödinger equations. The discovery of Berry's phase is not only a breakthrough in the older theory of quantum adiabatic approximations, but also provides us with new insights in many physical phenomena. The concept of Berry's phase has developed in some different directions [15–26]. In this letter, by using the Lewis–Riesenfeld invariant theory, we shall study the dynamical and the geometric phases in a generalized time-dependent k-photon Λ-type Jaynes–Cummings model.

2 Model

The Hamiltonian of the simplest generalized time-dependent k-photon Jaynes–Cummings model for three-level atoms interacting with light field can be written as

$$\hat{H} = \omega(t)\hat{a}^\dagger\hat{a} + \omega_0(t)\hat{S}_z + \sqrt{2}g(t)[\hat{a}^k\hat{S}_+ + (\hat{a}^\dagger)^k\hat{S}_-], \tag{1}$$

where $\hat{S}_z$ and $\hat{S}_\pm$ are [27–29]

$$\hat{S}_+ = \frac{1}{\sqrt{2}}|3\rangle(\langle 1| + \langle 2|), \qquad \hat{S}_- = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)\langle 3|, \tag{2}$$

$$\hat{S}_z = \frac{1}{2}[|3\rangle\langle 3| - \frac{1}{2}(|1\rangle + |2\rangle)(\langle 1| + \langle 2|)], \tag{3}$$

satisfying $[\hat{S}_+, \hat{S}_-] = 2\hat{S}_z$ and $[\hat{S}_z, \hat{S}_\pm] = \pm\hat{S}_\pm$. This model describes a Λ-type k-photon transition of a three-level atom whose two lower levels are nearly degenerate. $\hat{a}(\hat{a}^\dagger)$ are the photon annihilation (creation) operators, $\omega(t)$ is the frequency of the photon field, $g(t)$ is the coupling coefficient between photons and atoms, and $\omega_0(t)$ is the atom transition frequency. $\hat{S}_+$, $\hat{S}_-$, and $\hat{S}_z$ are defined as

$$\hat{S}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{S}_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \hat{S}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \tag{4}$$

If introducing the supersymmetric generators $\hat{Q}_+ = \hat{a}^k\hat{S}_+$ and $\hat{Q}_- = (\hat{a}^\dagger)^k\hat{S}_-$, one has

$$\{\hat{Q}_+, \hat{Q}_-\} = \frac{1}{2}\begin{pmatrix} 2\hat{a}^k\hat{a}^{\dagger k} & 0 & 0 \\ 0 & \hat{a}^{\dagger k}\hat{a}^k & \hat{a}^{\dagger k}\hat{a}^k \\ 0 & \hat{a}^{\dagger k}\hat{a}^k & \hat{a}^{\dagger k}\hat{a}^k \end{pmatrix} \equiv \hat{\hat{N}}. \tag{5}$$

We can verify that Hamiltonian (1) is solvable and the complete set of exact solutions can be found by working in a sub-Hilbert space corresponding to a particular eigenvalue of the supersymmetric generator $\hat{\hat{N}}$. It is easy to find that $\hat{\hat{N}}$, $\hat{a}^\dagger\hat{a}$, $\hat{Q}_+$, $\hat{Q}_-$, $\hat{S}_+$, $\hat{S}_-$, and $\hat{S}_z$ are the supersymmetric generators and they form the supersymmetric Lie algebra, namely

$$\hat{Q}_-^2 = \hat{Q}_+^2 = 0, \qquad [\hat{Q}_+, \hat{Q}_-] = 2\hat{\hat{N}}\hat{S}_z, \qquad [\hat{\hat{N}}, \hat{Q}_-] = [\hat{\hat{N}}, \hat{Q}_+] = [\hat{\hat{N}}, \hat{S}_z] = 0, \tag{6}$$

$$[\hat{\hat{N}}, \hat{a}^\dagger \hat{a}] = 0, \qquad \{\hat{Q}_-, \hat{S}_z\} = \{\hat{Q}_+, \hat{S}_z\} = 0, \qquad [\hat{S}_z, \hat{a}^\dagger \hat{a}] = 0, \tag{7}$$

$$\hat{S}_z(\hat{Q}_+ - \hat{Q}_-) = \frac{1}{2}(\hat{Q}_+ + \hat{Q}_-), \qquad (\hat{Q}_+ - \hat{Q}_-)^2 = -\hat{\hat{N}}, \qquad [\hat{Q}_-, \hat{a}^\dagger \hat{a}] = -k\hat{Q}_-, \tag{8}$$

$$[\hat{Q}_+, \hat{a}^\dagger \hat{a}] = k\hat{Q}_+, \qquad [\hat{Q}_-, \hat{S}_z] = \hat{Q}_-, \qquad [\hat{Q}_+, \hat{S}_z] = -\hat{Q}_+, \tag{9}$$

where {, } stands for the anticommuting bracket. Correspondingly, (1) becomes

$$\hat{H} = \omega(t)\hat{a}^\dagger \hat{a} + \omega_0(t)\hat{S}_z + \sqrt{2}g(t)(\hat{Q}_+ + \hat{Q}_-). \tag{10}$$

It is easy to find that

$$\hat{\hat{N}} \begin{pmatrix} |n\rangle \\ 0 \\ 0 \end{pmatrix} = \frac{(n+k)!}{n!} \begin{pmatrix} |n\rangle \\ 0 \\ 0 \end{pmatrix}, \tag{11}$$

$$\hat{\hat{N}} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ |n+1\rangle \\ |n+1\rangle \end{pmatrix} = \frac{(n+1)!}{(n-k+1)!} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ |n+1\rangle \\ |n+1\rangle \end{pmatrix}, \tag{12}$$

$$\hat{\hat{N}} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ |n+1\rangle \\ -|n+1\rangle \end{pmatrix} = 0, \tag{13}$$

so we can restrict our study in the sub-Hilbert space of the supersymmetric quasi-algebra constructed by operators $\hat{\hat{N}}$, $\hat{a}^\dagger \hat{a}$, $\hat{Q}_+$, $\hat{Q}_-$, $\hat{S}_+$, $\hat{S}_-$, and $\hat{S}_z$. Below, we will replace operator $\hat{\hat{N}}$ with the particular eigenvalues $\frac{(n+k)!}{n!}$, $\frac{(n+1)!}{(n-k+1)!}$, and 0, respectively.

3 Dynamical and Geometric Phases

We first illustrate the Lewis–Riesenfeld (L–R) invariant theory [10]. For a one-dimensional system whose Hamiltonian $\hat{H}(t)$ is time-dependent, then there exists an operator $\hat{I}(t)$ called invariant if it satisfies the equation

$$i\frac{\partial \hat{I}(t)}{\partial t} + [\hat{I}(t), \hat{H}(t)] = 0. \tag{14}$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given

$$\hat{I}(t)|\lambda_n, t\rangle = \lambda_n |\lambda_n, t\rangle, \tag{15}$$

where $\frac{\partial \lambda_n}{\partial t} = 0$. The time-dependent Schrödinger equation for this system is

$$i\frac{\partial |\psi(t)\rangle_s}{\partial t} = \hat{H}(t)|\psi(t)\rangle_s. \tag{16}$$

According to the L–R invariant theory, the particular solution $|\lambda_n, t>_s$ of (16) is different from the eigenfunction $|\lambda_n, t\rangle$ of $\hat{I}(t)$ only by a phase factor $\exp[i\delta_n(t)]$, i.e.,

$$|\lambda_n, t\rangle_s = \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{17}$$

which shows that $|\lambda_n, t\rangle_s$ $(n = 1, 2, \ldots)$ forms a complete set of the solutions of (16). Then the general solution of the Schrödinger equation (16) can be written by

$$|\psi(t)\rangle_s = \sum_n C_n \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{18}$$

where $C_n = \langle\lambda_n, 0|\psi(0)\rangle_s$, and the phase $\delta_n(t)$ is determined by

$$\delta_n(t) = \int_0^t dt'\langle\lambda_n, t'|i\frac{\partial}{\partial t'} - \hat{H}(t')|\lambda_n, t'\rangle. \tag{19}$$

For the system described by the Hamiltonian (10), we can define the following invariant

$$\hat{I}(t) = \alpha(t)\hat{Q}_- + \alpha^*(t)\hat{Q}_+ + \beta(t)\hat{S}_z. \tag{20}$$

Substituting (10) and (20) into (14), one has the auxiliary equations

$$i\dot{\alpha}(t) + [\omega_0(t) - k\omega(t)]\alpha(t) - \sqrt{2}g(t)\beta(t) = 0, \tag{21}$$

$$i\dot{\alpha}^*(t) + [k\omega(t) - \omega_0(t)]\alpha^*(t) + \sqrt{2}g(t)\beta(t) = 0, \tag{22}$$

$$i\dot{\beta}(t) + 2\sqrt{2}g(t)\bar{N}[\alpha^*(t) - \alpha(t)] = 0, \tag{23}$$

where dot denotes the time derivative, and $\bar{N}$ denotes the eigenvalue of operator $\hat{\bar{N}}$.

In order to obtain a time-independent invariant, we can introduce the unitary transformation operator $\hat{V}(t) = \exp[\xi(t)\hat{Q}_- - \xi^*(t)\hat{Q}_+]$. It is easy to find that when satisfying the following relations

$$\sin\big(2\sqrt{\bar{N}}|\xi(t)|\big) = \frac{\sqrt{\bar{N}}[\alpha(t)\xi^*(t) + \alpha^*(t)\xi(t)]}{|\xi(t)|}, \qquad \beta(t) = \cos\big(2\sqrt{\bar{N}}|\xi(t)|\big), \tag{24}$$

$$\alpha(t) - \frac{\beta(t)\xi(t)}{2\sqrt{\bar{N}}|\xi(t)|}\sin\big(2\sqrt{\bar{N}}|\xi(t)|\big) + \frac{\xi(t)[\alpha(t)\xi^*(t) + \alpha^*(t)\xi(t)]}{2|\xi(t)|^2}\big[\cos(2\sqrt{\bar{N}}|\xi(t)|) - 1\big] = 0, \tag{25}$$

then a time-independent invariant appears

$$\hat{I}_V \equiv \hat{V}^\dagger(t)\hat{I}(t)\hat{V}(t) = \hat{S}_z. \tag{26}$$

According to (24) and (25), we can select

$$\xi(t) = \frac{\theta(t)}{2\sqrt{\bar{N}}}\exp[i\gamma(t)], \qquad \alpha(t) = \frac{\sin\theta(t)}{2\sqrt{\bar{N}}}\exp[i\gamma(t)], \quad \theta(t) = 2\sqrt{\bar{N}}|\xi(t)|. \tag{27}$$

From (27), the invariant $\hat{I}(t)$ in (20) becomes

$$\hat{I}(t) = \frac{\sin\theta(t)}{2\sqrt{\bar{N}}}\big\{\exp[i\gamma(t)]\hat{Q}_- + \exp[-i\gamma(t)]\hat{Q}_+\big\} + \cos\theta(t)\hat{S}_z. \tag{28}$$

By using the Baker–Campbell–Hausdorff formula [30]

$$\hat{V}^\dagger(t)\frac{\partial \hat{V}(t)}{\partial t}=\frac{\partial \hat{L}}{\partial t}+\frac{1}{2!}\left[\frac{\partial \hat{L}}{\partial t},\hat{L}\right]+\frac{1}{3!}\left[\left[\frac{\partial \hat{L}}{\partial t},\hat{L}\right],\hat{L}\right]+\frac{1}{4!}\left[\left[\left[\frac{\partial \hat{L}}{\partial t},\hat{L}\right],\hat{L}\right],\hat{L}\right]+\cdots, \tag{29}$$

with $\hat{V}(t)=\exp[\hat{L}(t)]$, it is easy to find that when satisfying the following equations

$$\begin{aligned}&[k\omega(t)-\omega_0(t)]\sin\theta(t)\cos\gamma(t)+g(t)\sqrt{2\bar{N}}\{2-[1-\cos\theta(t)][1+\cos 2\gamma(t)]\}\\&\quad+[\dot{\theta}(t)+\dot{\gamma}(t)\cos\gamma(t)]\sin\theta(t)=0,\end{aligned} \tag{30}$$

$$\begin{aligned}&[k\omega(t)-\omega_0(t)]\sin\theta(t)\sin\gamma(t)-g(t)\sqrt{2\bar{N}}[1-\cos\theta(t)]\sin 2\gamma(t)\\&\quad-\dot{\theta}(t)\cos\theta(t)+\dot{\gamma}(t)\sin\theta(t)\sin\gamma(t)=0,\end{aligned} \tag{31}$$

one has

$$\begin{aligned}\hat{H}_V(t)&=\hat{V}^\dagger(t)\hat{H}(t)\hat{V}(t)-i\hat{V}^\dagger(t)\frac{\partial \hat{V}(t)}{\partial t}\\&=\omega(t)\hat{a}^\dagger\hat{a}+\{\omega_0(t)\cos\theta(t)+k\omega(t)[1-\cos\theta(t)]\\&\quad+2\sqrt{2\bar{N}}g(t)\sin\theta(t)\cos\gamma(t)\}\hat{S}_z+\frac{\dot{\gamma}(t)}{2}[1-\cos\theta(t)]\hat{S}_z.\end{aligned} \tag{32}$$

The eigenstates of operator $\hat{S}_z$ corresponding to the eigenvalues $S_z=+1$, $S_z=-1$, and $S_z=0$ are

$$\begin{pmatrix}1\\0\\0\end{pmatrix},\qquad \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix},\quad\text{and}\quad \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix},$$

respectively. The eigenstates of operator $\hat{\bar{N}}$ are

$$\begin{pmatrix}|n\rangle\\0\\0\end{pmatrix},\qquad \frac{1}{\sqrt{2}}\begin{pmatrix}0\\|n+1\rangle\\|n+1\rangle\end{pmatrix},\quad\text{and}\quad \frac{1}{\sqrt{2}}\begin{pmatrix}0\\|n+1\rangle\\-|n+1\rangle\end{pmatrix},$$

respectively. From (11)–(13), we can obtain three particular solutions of the time-dependent Schrödinger equation (16), respectively. For $S_z=+1$, one has

$$|\psi_{S_z=+1}(t)\rangle=\exp\left\{-i\int_0^t\left[\dot{\delta}^d_{S_z=+1}(t')+\dot{\delta}^g_{S_z=+1}(t')\right]dt'\right\}\hat{V}(t)\begin{pmatrix}|n\rangle\\0\\0\end{pmatrix}, \tag{33}$$

where

$$\begin{aligned}\dot{\delta}^d_{S_z=+1}(t')&=n\omega(t')+\omega_0(t')\cos\theta(t')+k\omega(t')[1-\cos\theta(t')]\\&\quad+2\sqrt{\frac{2(n+k)!}{n!}}g(t')\sin\theta(t')\cos\gamma(t'),\end{aligned} \tag{34}$$

$$\dot{\delta}^g_{S_z=+1}(t')=\frac{\dot{\gamma}(t')}{2}[1-\cos\theta(t')], \tag{35}$$

for $S_z=-1$,

$$|\psi_{S_z=-1}(t)\rangle=\exp\left\{-i\int_0^t\left[\dot{\delta}^d_{S_z=-1}(t')+\dot{\delta}^g_{S_z=-1}(t')\right]dt'\right\}\hat{V}(t)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\|n+1\rangle\\|n+1\rangle\end{pmatrix}, \quad (36)$$

where

$$\dot{\delta}^d_{S_z=-1}(t')=\frac{n+1}{\sqrt{2}}\omega(t')+\frac{1}{\sqrt{2}}\omega_0(t')\cos\theta(t')+\frac{k}{\sqrt{2}}\omega(t')[1-\cos\theta(t')] + 2\sqrt{\frac{(n+1)!}{(n-k+1)!}}g(t')\sin\theta(t')\cos\gamma(t'), \quad (37)$$

$$\dot{\delta}^g_{S_z=-1}(t')=\frac{\dot{\gamma}(t')}{2}[1-\cos\theta(t')], \quad (38)$$

and for $S_z=0$,

$$|\psi_{S_z=0}(t)\rangle=\exp\left\{-i\int_0^t\left[\dot{\delta}^d_{S_z=0}(t')+\dot{\delta}^g_{S_z=0}(t')\right]dt'\right\}\hat{V}(t)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\|n+1\rangle\\-|n+1\rangle\end{pmatrix}, \quad (39)$$

where

$$\dot{\delta}^d_{S_z=0}(t')=-\frac{n+1}{\sqrt{2}}\omega(t')-\frac{1}{\sqrt{2}}\omega_0(t')\cos\theta(t')-\frac{k}{\sqrt{2}}\omega(t')[1-\cos\theta(t')], \quad (40)$$

$$\dot{\delta}^g_{S_z=0}(t')=-\frac{\dot{\gamma}(t')}{2}[1-\cos\theta(t')]. \quad (41)$$

From (34)–(35), (37)–(38), and (40)–(41), we conclude that the dynamical and the geometric phase factors of the system are $\exp[-i\int_0^t\dot{\delta}^d_{S_z}(t')dt']$ and $\exp[-i\int_0^t\dot{\delta}^g_{S_z}(t')dt']$ with $S_z=\pm1,0$, respectively. In particular, when we consider a cycle in the parameter space of the invariant $\hat{I}(t)$ and let $\theta(t)$=constant, one has from (35), (38) and (41), respectively

$$\delta^g_{S_z}(T)=\begin{cases}\frac{1}{2}2\pi(1-\cos\theta) & (S_z=+1),\\ \frac{1}{2}2\pi(1-\cos\theta) & (S_z=-1),\\ -\frac{1}{2}2\pi(1-\cos\theta) & (S_z=0).\end{cases} \quad (42)$$

Here $2\pi(1-\cos\theta)$ denotes the solid angle over the parameter space of the invariant $\hat{I}(t)$. It is of interest that $\pm\frac{1}{2}2\pi(1-\cos\theta)$ is equal to the magnetic flux produced by a monopole of strength $\frac{1}{8\pi}$ existing at the origin of the parameter space. It is pointed out that the Geometric Phases in the cycle case are independent of the photon numbers k, the frequency $\omega(t)$ of the photon field, the coupling coefficient $g(t)$ between photons and atoms, and the atom transition frequency $\omega_0(t)$. The geometric phase differs from the dynamical phase and it involves the global and topological properties of the time evolution of a quantum system.

4 Conclusions

In summary, by using the Lewis–Riesenfeld invariant theory, we have studied the dynamical and the Geometric Phases in a generalized time-dependent k-photon Λ-type Jaynes–Cummings model. We find that, different from the dynamical phases, the Geometric Phases in a cycle case are independent of the photon numbers k, the frequency $\omega(t)$ of the photon field, the coupling coefficient $g(t)$ between photons and atoms, and the atom transition frequency $\omega_0(t)$.

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